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OLYMPIAD PROBLEMS FROM ALL OVER THE WORLD

VOLUME 4
8th GRADE CONTENT

Cartea Românească
EDUCAȚIONAL

Table of Contents

Chapter I
Problems

1. Find all pairs (a, b) of non-negative integers such that $2017^a = b^6 - 32b + 1$.

WALTER JANOUS, AUSTRIAN NMO, 2017

2. Let $(a_n)_{n\geq 0}$ be the sequence of rational numbers with $a_0 = 2016$ and $a_{n+1} = a_n + \dfrac{2}{a_n}$ for all $n \geq 0$. Show that the sequence does not contain a square of a rational number.

THERESIA EISENKOLBL, AUSTRIAN NMO, 2017

3. a) Determine the maximum M of $x + y + z$ where x, y and z are positive real numbers with $16xyz = (x + y)^2(x + z)^2$.

b) Prove the existence of infinitely many triples (x, y, z) of positive rational numbers that satisfy $16xyz = (x + y)^2(x + z)^2$ and $x + y + z = M$.

CARL KZAKLER, AUSTRIAN NMO, 2017

4. Let $m > 2017$ be positive integer and $N = m^{2017} + 1$. The numbers N, $N - m$, $N - 2m$, …, $m + 1$, 1 are written (in that order) on the blackboard. On every move the left most number is deleted together with all its divisors (if any). Find the last deleted number.

ALEKSANDAR IVANOV, BULGARIAN NMO, 2017

5. Find all primes p and all positive integers a and m such that

$$a \leq 5p^2 \text{ and } (p-1)! + a = p^m.$$

MIROSLAV MARINOV, BULGARIAN NMO, 2017

6. Let $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n, p$ be real numbers with $p > -1$. Prove that:

$$\sum (a_i - b_i)\left(a_i\left(a_1^2 + a_2^2 + \dots + a_n^2\right)^{\frac{p}{2}} - b_i\left(b_1^2 + b_2^2 + \dots + b_n^2\right)^{\frac{p}{2}}\right) \geq 0.$$

SINGAPORE, SMO, 2017

7. Find the smallest positive integer n so that $\sqrt{\dfrac{1^2 + 2^2 + \dots + n^2}{n}}$ is an integer.

SINGAPORE, SMO, 2017

8. Let A and B be two $n \times n$ square arrays. The cells of A are labelled by the numbers from 1 to n^2 from left to right starting from the top row; whereas the cells of B are labelled by the numbers from 1 to n^2 along rising north-easterly diagonals starting with the upper left-band corner. Stack the array B on top of the array A. If two overlapping cells have the same number, they are coloured red. Determine those n for which there is at least one red cell other than the cells at top left corner, bottom right corner and the centre (when n is odd). Below shows the arrays for $n = 4$:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}; \; B = \begin{bmatrix} 1 & 3 & 6 & 10 \\ 2 & 5 & 9 & 13 \\ 4 & 8 & 12 & 15 \\ 7 & 11 & 14 & 16 \end{bmatrix}$$

SINGAPORE, SMO, 2017

9. Determine, with proof, the smallest positive multiple of 99 all of whose digits are either 1 or 2.

STEPHEN BUCKLEY, IRELAND SHL, 2017

10. Let a and b be positive integers that are co-prime and let p be a prime number. Prove that:

$$\gcd(ab, a^2 + pb^2) = \begin{cases} 1 \text{ if } p \nmid a \\ p \text{ if } p \mid a \end{cases}.$$

BERND KREUSSLER, IRELAND NMO, 2017

11. Find all pairs (t, x) of real numbers that satisfy:

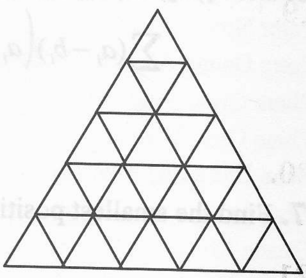
$$t^3 - 3t^2 + 3t - x = 0 \text{ and } 27(x-1)^4 + (1-x^2)^3 = 0.$$

FINBAR HOLLAND, IRELAND NMO, 2017

12. Does there exist an even positive integer n for which $n + 1$ is divisible by 5 and the two numbers $2^n + n$ and $2^n - 1$ are co-prime?

BERND KREUSSLER, IRELAND SHL, 2017

13. An equilateral triangle of integer side length $n \geq 1$ is subdivided into small triangles of unit side length, as illustrated in the figure below for the case $n = 5$. In this diagram, a *sub-triangle* is a triangle of any size which is formed by connecting vertices of the small triangles along the grid-lines. It is desired to colour each vertex of the small triangles either red or blue in such a way that there is no sub-triangle with all three of its vertices having the same colour. Let $f(n)$ denote the number of distinct colourings satisfying this condition.



Determine, with proof, $f(n)$ for every $n \geq 1$.

MARK FLANAGAN, IRELAND NMO, 2017

14. Show that for all non-negative numbers a, b,

$$1+a^{2017}+b^{2017}\geq a^{10}b^{7}+a^{7}b^{2000}+c^{2000}b^{10}.$$

When is equality attained?

STEPHEN BUCKLEY, IRELAND NMO, 2017

15. For which prime numbers p do there exist positive rational numbers x, y and a positive integer n such that $x+y+\dfrac{p}{x}+\dfrac{p}{y}=3n$?

BERND KREUSSLER, IRELAND SHL, 2017

16. If $a,b,c>0$, $a+b+c=abc$, then:

$$\frac{4(a+b)(a+c)}{(b+c)^2}+\frac{4(b+c)(b+a)}{(c+a)^2}+\frac{4(c+a)(c+b)}{(a+b)^2}\leq 3+a^2+b^2+c^2.$$

DANIEL SITARU, RMM, ROMANIA

17. If $x,y,z,t,a,b,c\in(0,\infty)$, $xyzt=a^4$ then:

$$3a\sum\frac{x^b+y^c}{y+z+t}\geq 4\left(a^b+a^c\right).$$

DANIEL SITARU, RMM, ROMANIA

18. If $M\in \mathrm{Int}(\Delta ABC)$, then:

$$MA+MB+MC\geq\frac{9s}{\sin A\cot^2\frac{C}{2}+\sin B\cot^2\frac{A}{2}+\sin C\cot^2\frac{B}{2}}.$$

DANIEL SITARU, RMM, ROMANIA

19. Find all x,y,z which satisfy $\begin{cases}x^2+xy+xz=y,\\ y^2+yz+yx=z,\\ z^2+zx+zy=x.\end{cases}$

BOGDAN RUBLYOV, UKRAINIAN NMO, 2017

20. Find all positive integers n, such that 11^n-1 is divisible by 10^n-1.

UKRAINIAN NMO, 2017

21. If a, b and c are the dimensions of a rectangular parallelepiped and d is its diagonal, prove that:

$$d\leq\sqrt{\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}}.$$

D.M. BĂTINEȚU-GIURGIU, NECULAI STANCIU, ROMANIA

22. The vertices of a cube are enumerated by the numbers 1; 2; ...; 8. Someone chose three sides of the cube and to id the numbers which are written on them to Pete: {1; 4; 6; 8}, {1; 2; 6; 7}, {1; 2; 5; 8}. Is it possible to determine which number has the vertex which is opposite to the one numbered 5?

UKRAINIAN NMO, 2016

23. Let a, b and c be the lengths of the edges of a rectangular parallelepiped. Prove that:

$$\frac{(a^n+b^n+c^n)(a^m+b^m+c^m)}{3}=a^{n+m}+b^{n+m}+c^{n+m},\ \forall n,m\in\mathbb{N},$$

if and only if the rectangular parallelepiped is a cube.

D.M. BĂTINEȚU-GIURGIU, NECULAI STANCIU, ROMANIA

24. Solve in the set $\mathbb{R}\times\mathbb{R}$ the system of equations:

$$\begin{cases} x^3+y^3=13 \\ x^2y+xy^2=-4 \end{cases}.$$

D.M. BĂTINEȚU-GIURGIU, NECULAI STANCIU, ROMANIA

25. Solve in $\mathbb{R}_+^*$ the system of equations:

$$\begin{cases} \dfrac{1}{x+y+x}=1 \\ \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}+\dfrac{1}{xyz}=36 \end{cases}.$$

ROXANA MIHAELA STANCIU, NECULAI STANCIU, ROMANIA

26. The twins Pete and Ostap had had an argue and started to go to school different ways. Pete goes 210 meters to the South, then 70 meters to the East and reaches the school. Ostap goes to the North for a while and then he head to the school directly. How many meters Ostap has to go to the North if both twins have equal speeds and come to the school.

UKRAINIAN NMO, 2016

27. Find all triplets (m, n, p) where m, n are two natural numbers and p is a prime number, satisfying the equation:

$$m^4=4\left(p^n-1\right).$$

NGUYEN VIET HUNG, VIETNAM, RMM AUTUMN EDITION, 2016

28. Prove that if $x, y, z>0$, $xyz=8$, then:

$$x^3+y^3+z^3\geq 2x\sqrt{y+z}+2y\sqrt{z+x}+2z\sqrt{x+y}.$$

IULIANA TRAȘCĂ, ROMANIA, RMM AUTUMN EDITION, 2016

29. If $x, y, z > 0$, then prove that:

$$\left(x^3y^3+y^3z^3+z^3x^3\right)\left(\frac{1}{(x+y)^5 z}+\frac{1}{(y+z)^5 x}+\frac{1}{(z+x)^5 y}\right)\geq\frac{9}{32}.$$

D.M. BĂTINEȚU-GIURGIU, NECULAI STANCIU, ROMANIA,
RMM AUTUMN EDITION, 2016

30. Determine the maximum number of queens that can be placed on a 2017 × 2017 chessboard so that each queen attacks at most one of the others.

BOJAN BASIK, SERBIAN NMO, 2017

31. Fourteen schools participate in the second Tha Sala Mathematics Talent competition, with each school sending 14 students. The students take tests in 14 rooms, with 14 students in a room such that every room does not contain students from the same school. Among the students there are 15 students who also participated in the First Tha Sala Mathematics Talent competition. At the opening ceremony the organisers will select 2 students from those who participated in the first competition to recite the pledge of honor, with the condition that the students are from different schools and take tests in different rooms. Let n be the number of ways to select 2 students satisfying the condition. Determine the least possible n.

THAILAND NMO, 2017

32. Find the minimum value of $\dfrac{a^3+b^3+c^3}{abc}$ when a, b and c are sides of a right triangle.

THAILAND NMO, 2017

33. A point (x, y) in the plane is a *lattice* point if x and y are both integers. Let n be a positive integer. Prove that there exists a disk in the plane containing exactly n lattice points in its interior.

THAILAND NMO, 2017

34. Find all primes p for which there exists a positive integer n such that $p^n + 1$ is a cube of a positive integer.

JAN MAZAK, ROBERT TOTH, CZECH & SLOVAK NMO, 2017

35. We have n^2 empty boxes, each of them having square base. The height and the width of each box belongs to $\{1, 2, \ldots, n\}$ and every two boxes differ in at least one of these two dimensions. One box fits into another one if both its dimensions are smaller and at least one is smaller by at least 2. In this way, we can form sequences of boxes (the first one in the second one, the second one in the third one, and so on). We put any such set of boxes on a different shelf. How many shelves do we need to store all the boxes?

PETER NOVOTNY, CZECH & SLOVAK NMO, 2017

36. Find the smallest positive integer that can be inserted between numbers 20 and 16 so that the resulting number 20...16 is a multiple of 2016.

RADEK HORENSKY, CZECH & SLOVAK NMO, 2017

37. Find all positive integers n with the following property: Numbers 1, 2, ..., n can be split into three disjoint non-empty subsets with mutually different sizes such that, for any pair of subsets, the subset with fewer elements has larger sum of its elements. (A size of a subset is the number of its elements.)

MARTIN PANAK, CZECH & SLOVAK NMO, 2017

38. If $x, y, z > 0$, then:

$$45 < \sum \frac{(3x+5y)(5x+3y)}{(x+y)^2} \le 48.$$

DANIEL SITARU, RMM, ROMANIA

39. If $a, b, c > 0$, $\sqrt{1+a^2} + \sqrt{1+b^2} + \sqrt{1+c^2} = 3\sqrt{2}$, then:

$$\frac{a}{\sqrt{1+a^2}} + \frac{b}{\sqrt{1+b^2}} + \frac{c}{\sqrt{1+c^2}} \le \frac{3\sqrt{2}}{2}.$$

DANIEL SITARU, RMM, ROMANIA

40. If $a, b, c > 0$, $a + b + c = 3$, then:

$$\sum \left(\sqrt{a(a+2b)} + \sqrt{b(b+2a)} \right) \le 6\sqrt{3}.$$

DANIEL SITARU, RMM, ROMANIA

41. Solve in R^2 the system of equations:

$$(x+\sqrt{x^2+1})(y+\sqrt{y^2+1}) = 2012;$$

$$x + y = \frac{2011}{\sqrt{2012}}.$$

D.M. BĂTINEȚU-GIURGIU, NECULAI STANCIU, ROMANIA

42. If $a, b, c > 0, x + y + z = 1$, then:

$$\sum (x+y)^2 \ge 4\sqrt{3xyz}.$$

DANIEL SITARU, RMM, ROMANIA

43. Prove that if $a, b, c > 0$, then:

$$\sqrt{\frac{a}{b+c}} + 2\sqrt{\frac{b}{c+a}} + 4\sqrt{\frac{c}{a+b}} \le \sqrt{7\left(\frac{a}{b+c} + \frac{2b}{c+a} + \frac{4c}{a+b}\right)}.$$

DANIEL SITARU, RMM, ROMANIA

44. If $a, b, c > 0$, $a + b + c = 3$, then:

$$\sum \frac{a^5 + a - 1}{a^3 + a^2 - 1} \geq ab + bc + ca.$$

DANIEL SITARU, RMM, ROMANIA

45. If $a, b, c > 0$, $a^2 + b^2 + c^2 = 3$, then:

$$\frac{a^3b^2}{a^2 + a + 1} + \frac{b^3c^2}{b^2 + b + 1} + \frac{c^3a^2}{c^2 + c + 1} < 1 + ab^2 + bc^2 + ca^2.$$

DANIEL SITARU, RMM, ROMANIA

46. If $x, y, z > 0$, $x + y + z = 1$, then:

$$\frac{x^2}{z} + \frac{y^2}{x} + \frac{z^2}{y} + 2\left(\frac{xy}{2} + \frac{yz}{x} + \frac{zx}{y}\right) \geq 3.$$

DANIEL SITARU, RMM, ROMANIA

47. If $x, y, z > 0$, then:

$$x^2y^2z^2\left(\frac{x+y}{x^5+y^5} + \frac{y+z}{y^5+z^5} + \frac{z+x}{z^5+x^5}\right) \leq x^2 + y^2 + z^2.$$

DANIEL SITARU, RMM, ROMANIA

48. If $x, y, z > 0$, $xyz = 1$, then:

$$\sum \left(x^2z - 1 + y^2z\right)\left(x^5 + y^5\right) \geq 6.$$

DANIEL SITARU, RMM, ROMANIA

49. If $a, b, c > 0$, then:

$$\frac{ba^3}{1 + a + a^2} + \frac{cb^3}{1 + b + b^2} + \frac{ac^3}{1 + c + c^2} < 1 + ab + bc + ca.$$

DANIEL SITARU, RMM, ROMANIA

50. Let $n \geq 2$ be an integer. A game is played on a $n \times n$ board by two players X and Y as follows.

- Order: The two players take turns to play. X plays in the 1st round, Y plays the 2nd round, then X plays the 3rd round and so on.
- Rule: In the kth round, one has to choose k unmarked consecutive cells in the same row or in the same column and mark each of these cells.
- Winner: The first player who cannot complete the task will lose the game.

The player who is expected to carry out the $(n + 1)$st round is called the *natural loser*, as there are no $(n + 1)$ consecutive cells on the board. Find the smallest n for which the natural loser has a winning strategy.

HONG KONG, PREIMO 2017, MOCK EXAM

51. Let m and n be two integers and define $a_0 = m$, $a_1 = n$ and $a_{k+1} = 4a_k - 5a_{k-1}$ for $k \geq 1$. If $p > 5$ be a prime such that $p - 1$ is divisible by 4, then show that there are integers m and n such that p does not divide a_k for any $k \geq 0$.

INDIA TST, 2017

52. Suppose $n \geq 0$ is an integer and all the roots of $x^3 + \alpha x + 4 - (2 \times 2016^n) = 0$ are integers. Find all possible values of α.

INDIAN NMO, 2017

53. Find the number of triples (x, a, b) where x is a real number and a, b belong to the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that:

$$x^2 - a\{x\} + b = 0,$$

where $\{x\}$ denotes the fractional part of the real number x. (For example $\{1.1\} = 0.1 = \{-0.9\}$.)

INDIAN NMO, 2017

54. If $a, b, c > 0$ and $a + b + c = 3$, prove that:

$$\sum a\left(\frac{1}{b^n} + \frac{1}{c^n}\right) \geq \frac{18}{a^n + b^n + c^n}, \text{ where } n \geq 0.$$

MARIN CHIRCIU, RMM SPRING EDITION, 2017

55. Let a, b, c be positive real numbers. Prove that:

$$\frac{\left(a^2 - ab + b^2\right)^2}{(a+b)^4} + \frac{\left(b^2 - bc + c^2\right)^2}{(b+c)^4} + \frac{\left(c^2 - ca + a^2\right)^2}{(c+a)^4} \geq \frac{3}{16}.$$

GEORGE APOSTOLOPOULOS, RMM SPRING EDITION, 2017

56. Given an ordered pair of positive integers (x, y) with exactly one even coordinate, a *step* maps this pair to $\left(\frac{x}{2}, y + \frac{x}{2}\right)$ if $2 \mid x$ and to $\left(x + \frac{y}{2}, \frac{y}{2}\right)$ if $2 \mid y$. Prove that for every odd positive integer $n > 1$ there exists an even positive integer b, $b < n$, such that after finitely many steps the pair (n, b) maps to the pair (b, n).

BOJAN BASIK, SERBIAN TST, 2017

57. Let $n \geq 2$ be an integer. *Killer* is a game played by a dealer and n players. The game begins with the dealer designating one of the n players a killer and keeping this information a secret. Every player knows that the killer exists among the n players. The dealer can make as many public announcements as he wishes. Then, he secretly gives each of the n players a (possibly different) name of one of the n players. This game has the property that:

(i) Alone, each player (killer included) does not know who the killer is. Each player also cannot tell with certainty who is not the killer.

(ii) If any two of the n players exchange information, they can determine the killer. For example, if there are a dealer and 2 players, the dealer can announce that he will give the same name to both players if the first player is the killer, and give different names to the players if the second player is the killer.

a) Prove that Killer can be played with a dealer and 5 players.

b) Determine whether Killer can be played with a dealer and 4 players.

THAILAND NMO, 2017

58. Let $a_1 < a_2 < \ldots < a_{53}$ be positive integers such that the sum of any 27 integers is greater than the sum of the remaining 26 integers.

a) Find the minimum value of a_1.

b) Find the possible values of $a_2, \ldots, a_{53}$, when a_1 is at the minimum value.

TS. BATKHUU, MONGOLIAN NMO, 2017

59. Let $a_1 < a_2 < \ldots$ be the positive divisors of a positive integer a and let $b_1 < b_2 < \ldots$ be the positive divisors of a positive integer b. Find all a, b such that:

$$\begin{cases} a_{10} + b_{10} = a \\ a_{11} + b_{11} = b \end{cases}.$$

B. BATTSENGEL, MONGOLIAN NMO, 2017

60. Let a, b, c, d be positive real numbers such that $a + b + c + d = 4$. Prove that:

$$a\sqrt{a+8} + b\sqrt{b+8} + c\sqrt{c+8} + c\sqrt{c+8} \geq 12.$$

V. ADIYASUREN, MONGOLIAN NMO, 2017

61. Let a, b, c be positive real numbers such that $abc = 1$. Prove that:

$$\frac{1}{a^4+4b^4+7} + \frac{1}{b^4+4c^4+7} + \frac{1}{c^4+4a^4+7} \leq \frac{1}{4}.$$

T. BAZAR, MONGOLIAN NMO, 2017

62. Fix an integer $n \geq 2$ and positive reals $a < b$. Let $x_1, x_2, \ldots, x_n$ be real numbers in the closed interval $[a, b]$. Find the maximum of the following expression:

$$\frac{\dfrac{x_1^2}{x_2} + \dfrac{x_2^2}{x_3} + \ldots + \dfrac{x_{n-1}^2}{x_n} + \dfrac{x_n^2}{x_1}}{x_1 + x_2 + \ldots + x_{n-1} + x_n}.$$

CHINA NMO, 2017

63. Prove that if $a, b, c \in \mathbb{R}$, then:

$$(2 - a - b - c + abc)^2 \leq (a^2+2)(b^2+2)(c^2+2).$$

DANIEL SITARU, RMM, SUMMER EDITION, 2017

64. If $a, b, c > 0, n \geq 1$, then:

$$\frac{3n\left(a^4+b^4+c^4\right)}{\left(a^2+b^2+c^2\right)^2}+\frac{ab+bc+ca}{a^2+b^2+c^2} \geq n+1.$$

MARIN CHIRCIU, RMM, AUTUMN 2017

65. If $x, y, z > 0$, then:

$$\sqrt{\frac{13x}{6x+7y}}+\sqrt{\frac{13y}{6y+7z}}+\sqrt{\frac{13z}{6z+7x}} \leq 3.$$

MARIN CHIRCIU, RMM, AUTUMN EDITION, 2017

66. Let a and b be integers of different parity. Prove that there exists an integer c such that the numbers $ab + c$, $a + c$ and $b + c$ are squares of integers.

CROATIAN NMO, 2017

67. If x, y, z and w are real numbers such that:

$$\frac{x}{y+z+w}+\frac{y}{z+w+x}+\frac{z}{w+x+y}+\frac{w}{x+y+z}=1,$$

find
$$\frac{x^2}{y+z+w}+\frac{y^2}{z+w+x}+\frac{z^2}{w+x+y}+\frac{w^2}{x+y+z}.$$

CROATIAN NMO, 2017

68. We call a point P inside a triangle ABC *marvelous* if exactly 27 rays can be drawn from it, intersecting the sides of ABC such that the triangle is divided into 27 smaller triangles of equal areas. Determine the total number of marvellous points inside a given triangle ABC.

NABOJ, CROATIAN NMO, 2017

69. Let a, b, c be non-negative such that $a + b + c = 3$. Prove that:

$$|(a-b)(b-c)(c-a)| \leq \frac{3\sqrt{3}}{2}.$$

Equality occurs when?

NGUYEN NGOC TU, RMM, WINTER EDITION, 2017

70. Let m,n be positive real numbers. Prove that:

$$\left(\frac{1}{m}+\frac{1}{n}\right)^{-1} \leq \frac{4034-2015m}{m+2017}+\frac{4034-2015n}{n+2017}+\frac{m+n+2009}{2}.$$

IULIANA TRAȘCĂ, RMM, WINTER EDITION, 2017